Sensor network localization has benign landscape under mild rank relaxation

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with

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Run local algorithm (gradient descent, trust regions, …)

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Stable manifold theorems
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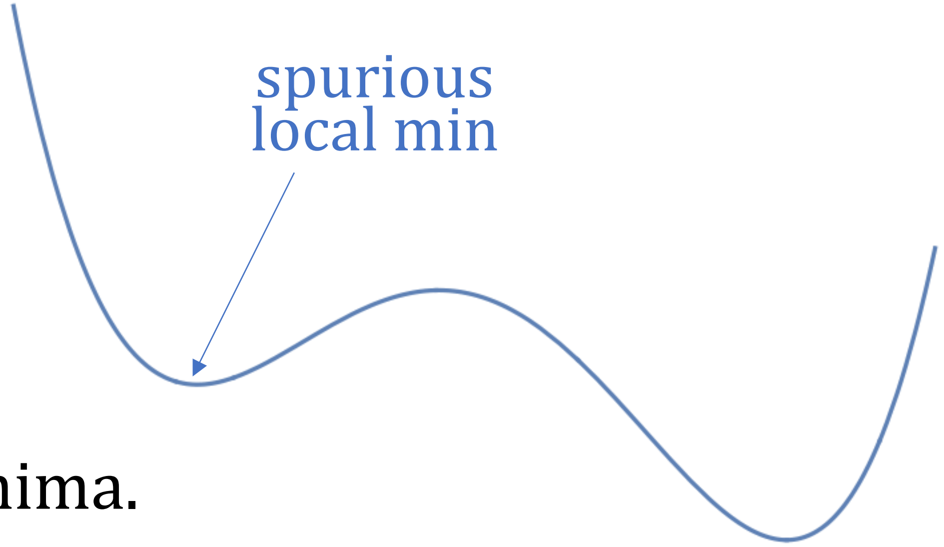

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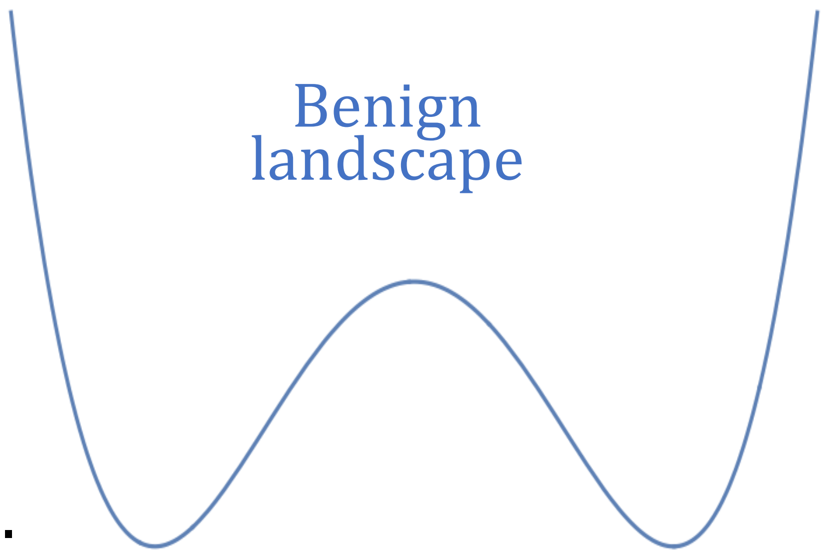

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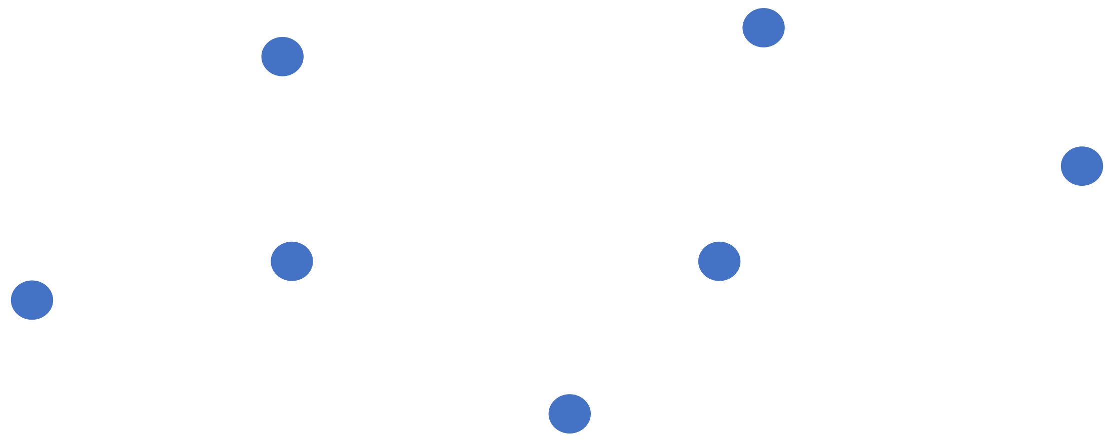
The problem

n unknown points $z_1^*, z_2^*, \ldots, z_n^*$ in $\mathbb{R}^\ell$.



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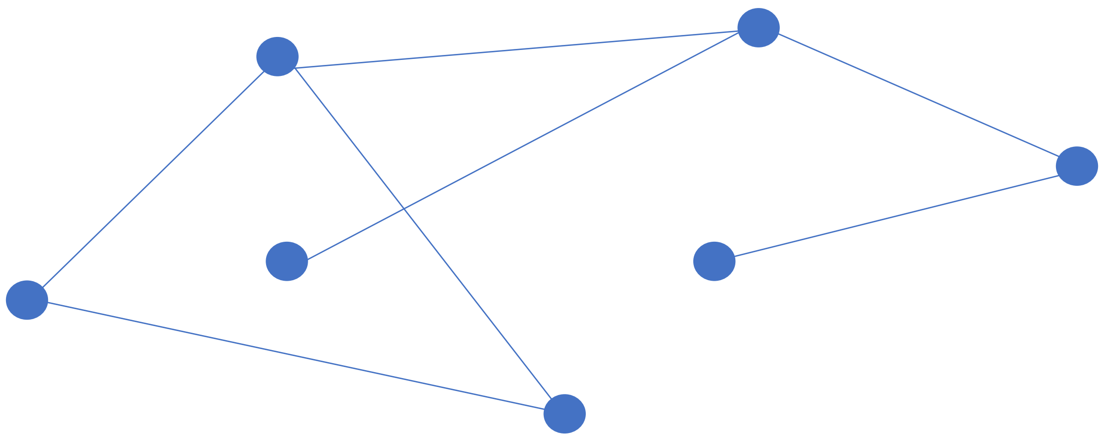

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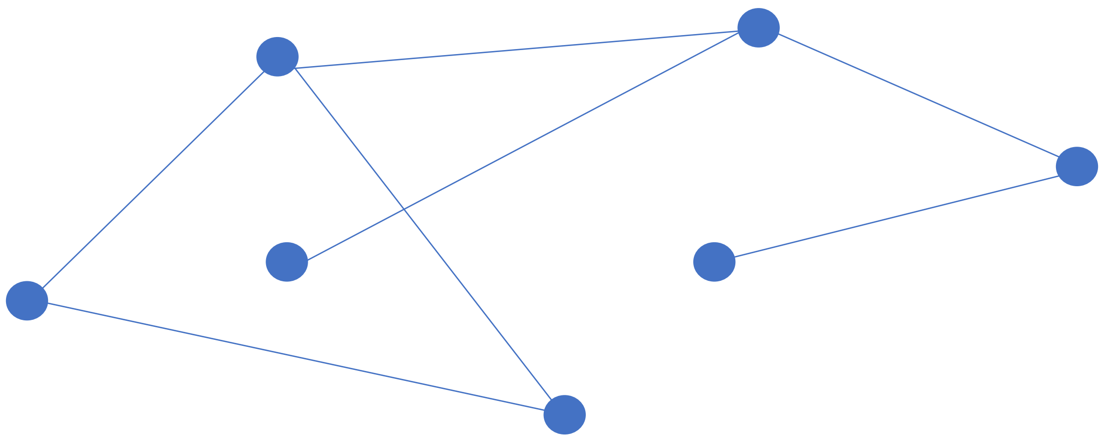
Goal: recover the n points (up to translation & rotation)



Applications

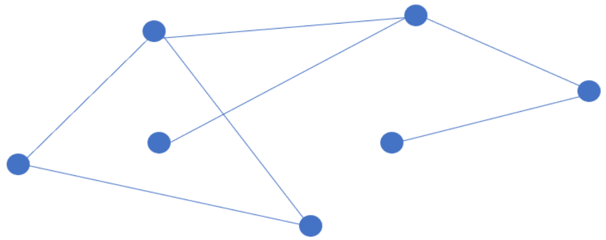
Robotics (**sensor network localization**), $\ell = 2,3$

Molecular conformation

Data analysis (metric **multidimensional scaling**)

Graph theory (rigidity)

When is recovery possible?



Global rigidity: Configuration space

$$\{z_1, z_2, \dots, z_n \in \mathbb{R}^\ell : d_{ij} = \|z_i - z_j\|\}$$

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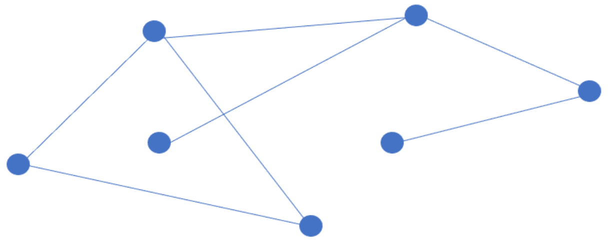

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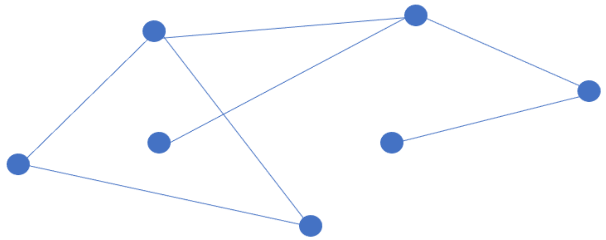

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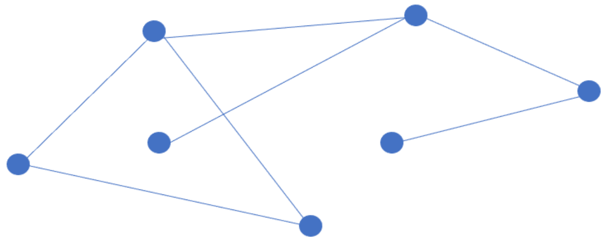

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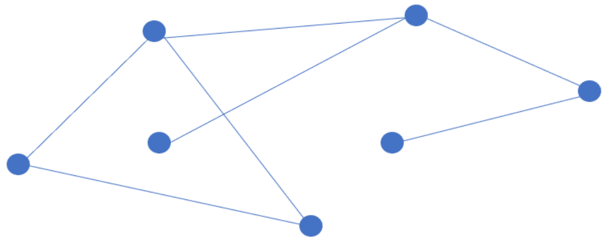

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Drawback: SDP involves $(n + \ell)\times(n + \ell)$ matrices

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Possible variations: Noisy measurements, landmarks, …

Our focus: (nearly) complete graphs, no noise

Synthetic experiments, complete graph

Recipe (all distances known):

(1) Choose ground truths $z_1^*, z_2^*, \ldots, z_n^*$ at random (normal iid)

(2) Run gradient descent/trust regions/etc.

(3) Find global min?

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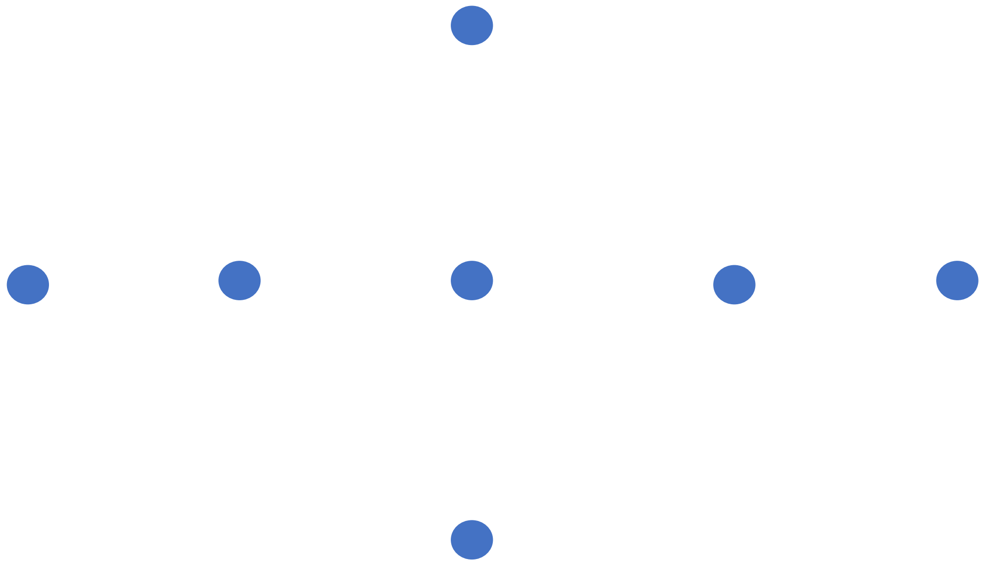
Open Question: Does s-stress have spurious local minima? Are all 2-critical points global minima?

* Malone & Trosset 2000, Parhizkar 2013, etc.

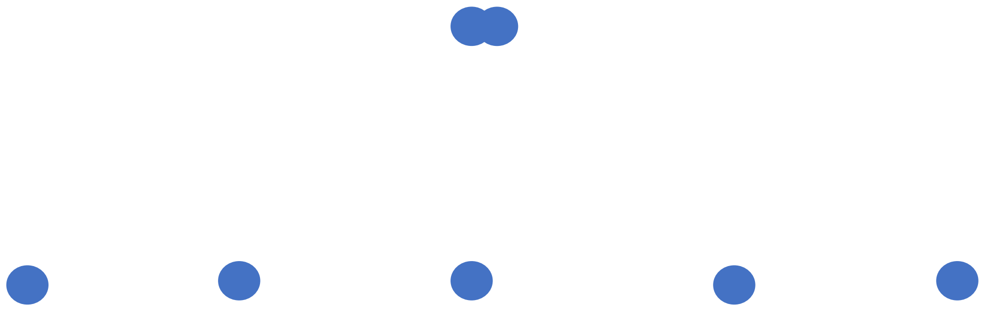
Counterexamples

s-stress can have spurious strict local minima!

Ground truth $z_1^*, z_2^*, \ldots$



Spurious configuration $z_1, z_2, \ldots$



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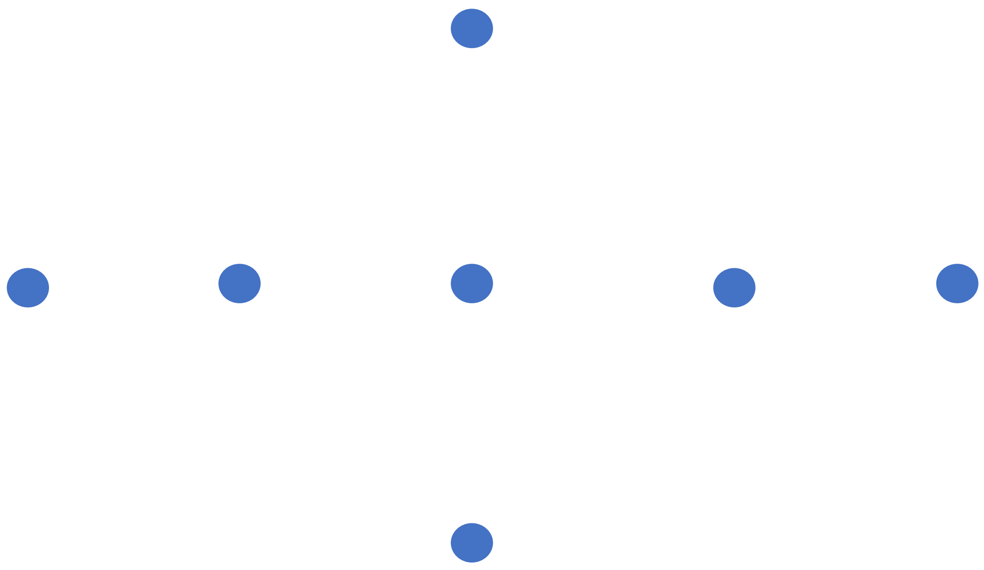

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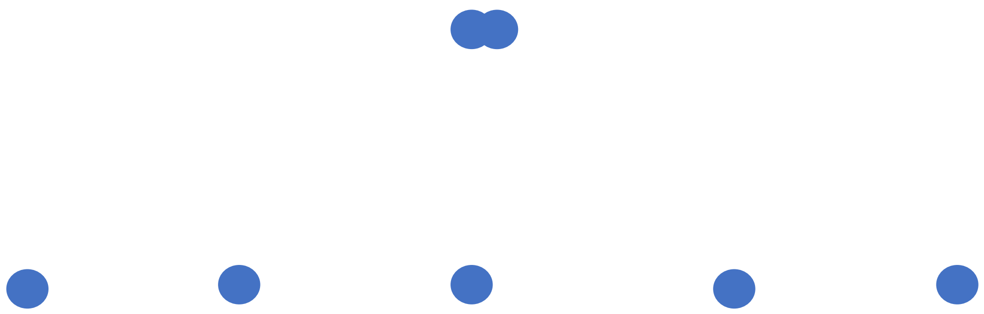

Also see:
Song, Goncalves, Jung, Lavor, Mucherino, Wolkowicz, 2024

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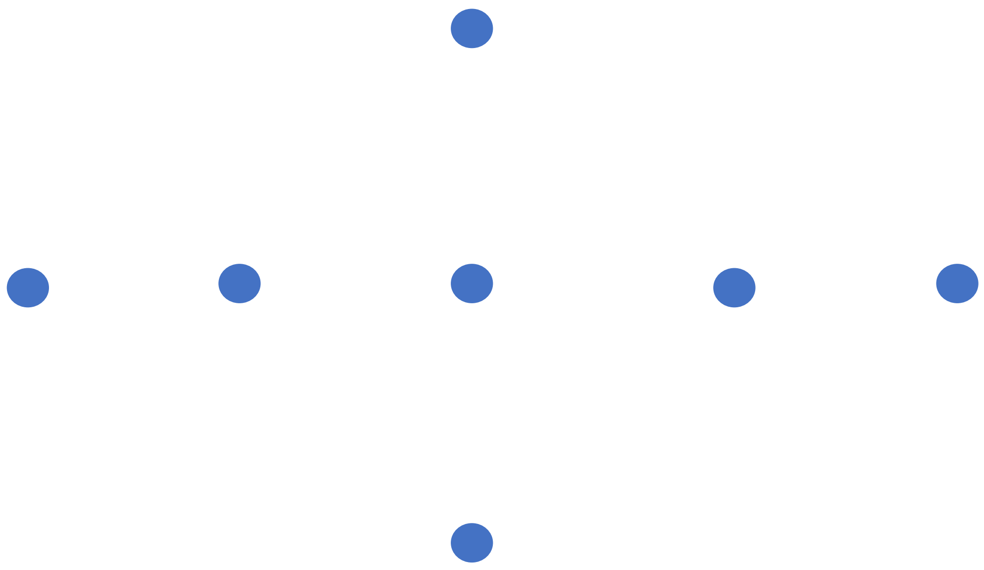

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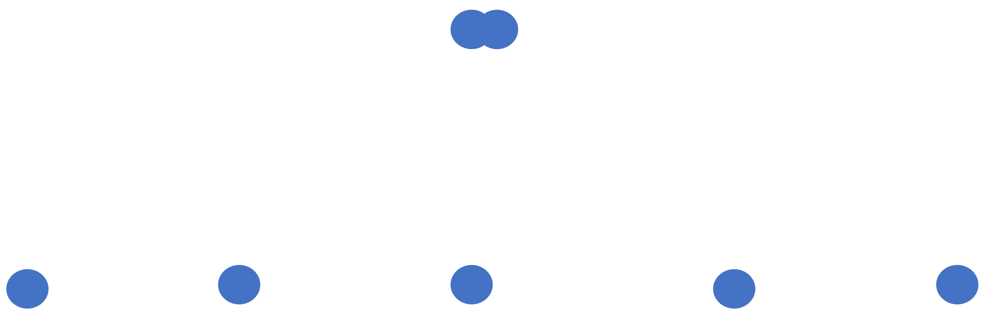

Set of ground truths with spurious local minima has positive measure

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Landscape is not benign, so we have to do something!
What?

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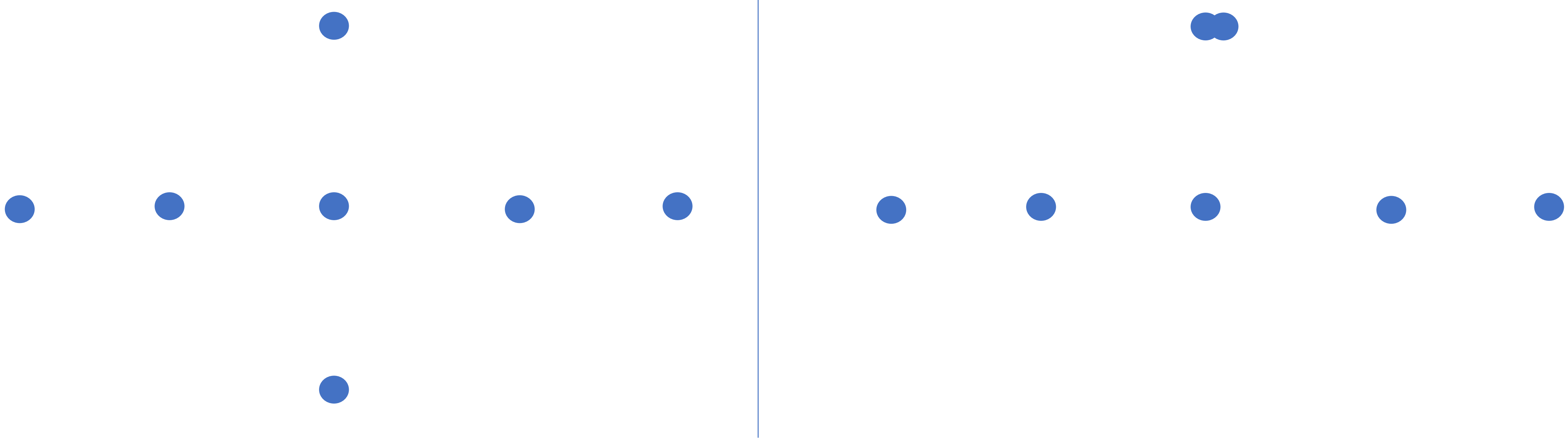

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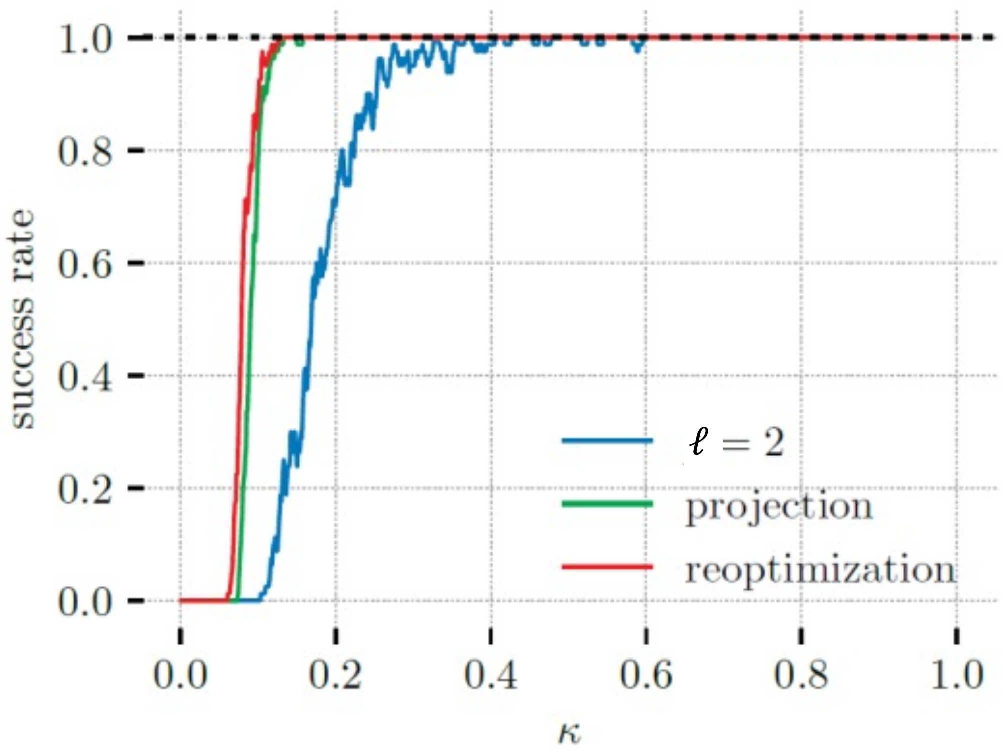

$n = 100$
$\ell = 2$
$k = 4$

$\kappa =$ edge density (Erdos-Renyi)

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If $k = n - 1$, easy to see landscape is benign (Song, Goncalves, Jung, Lavor, Mucherino, Wolkowicz, 2024)
Can we do better?

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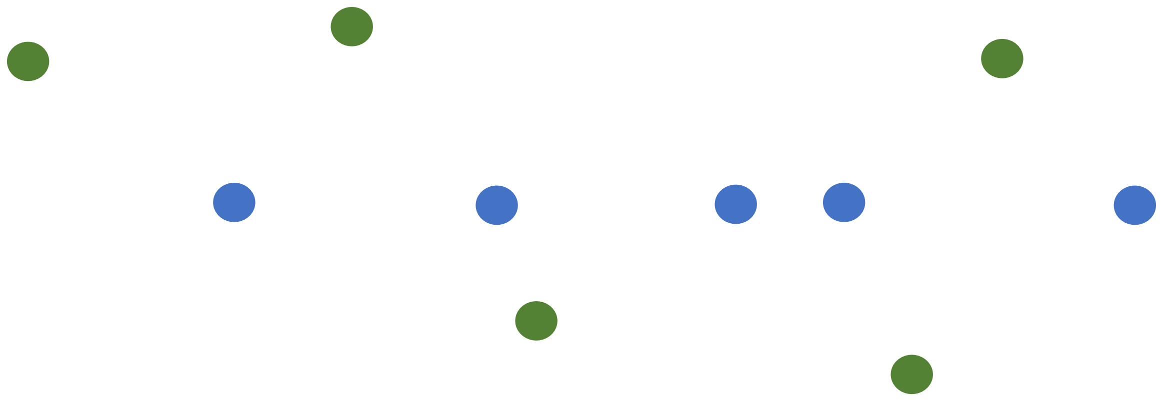

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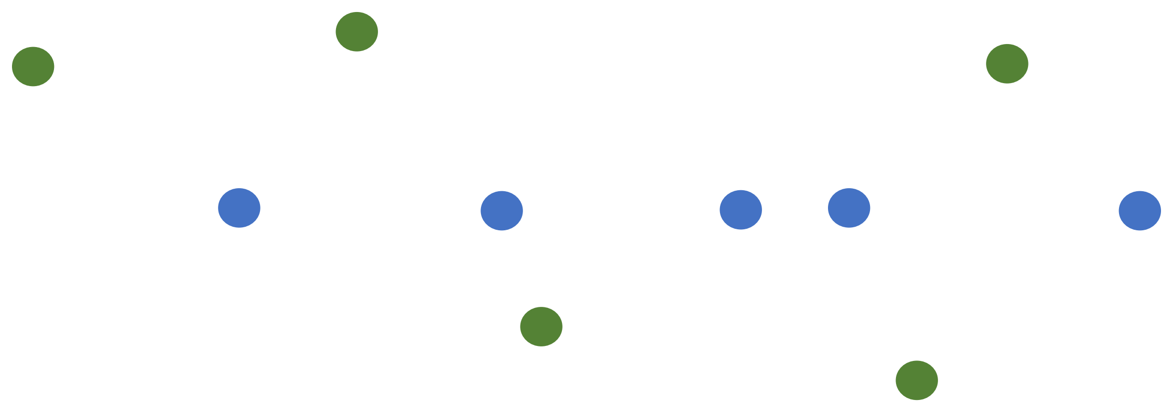

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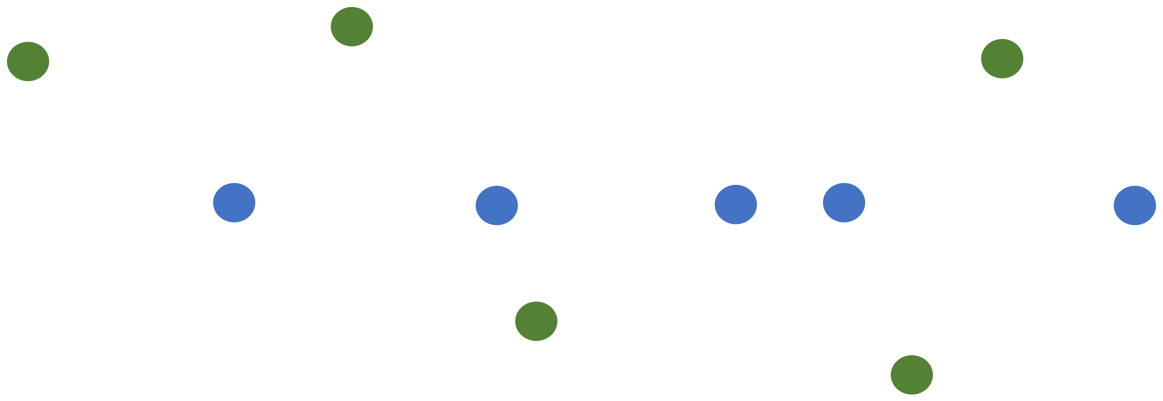

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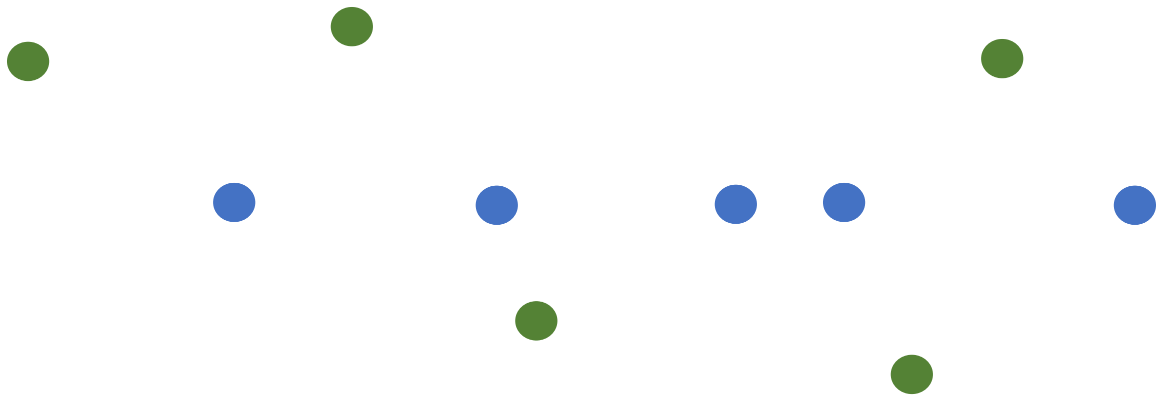

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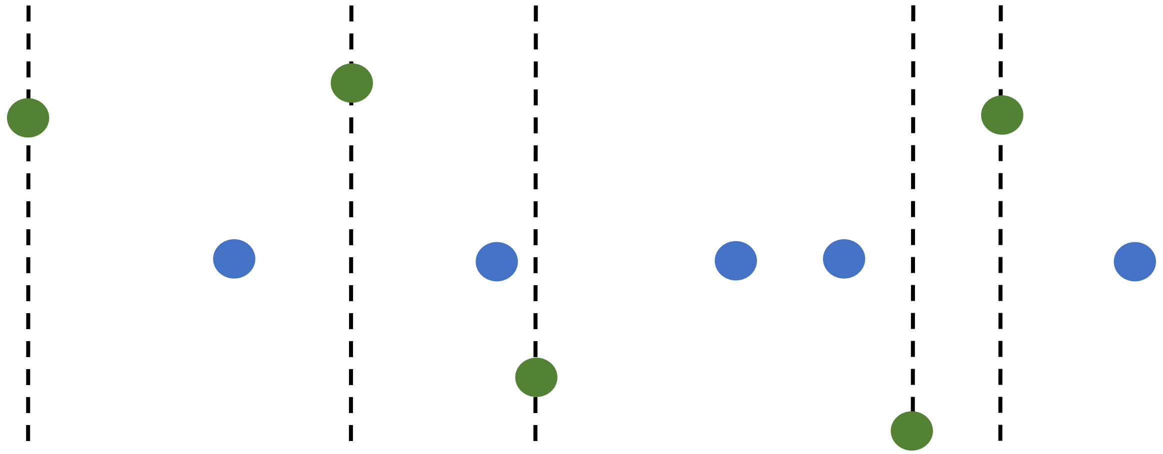

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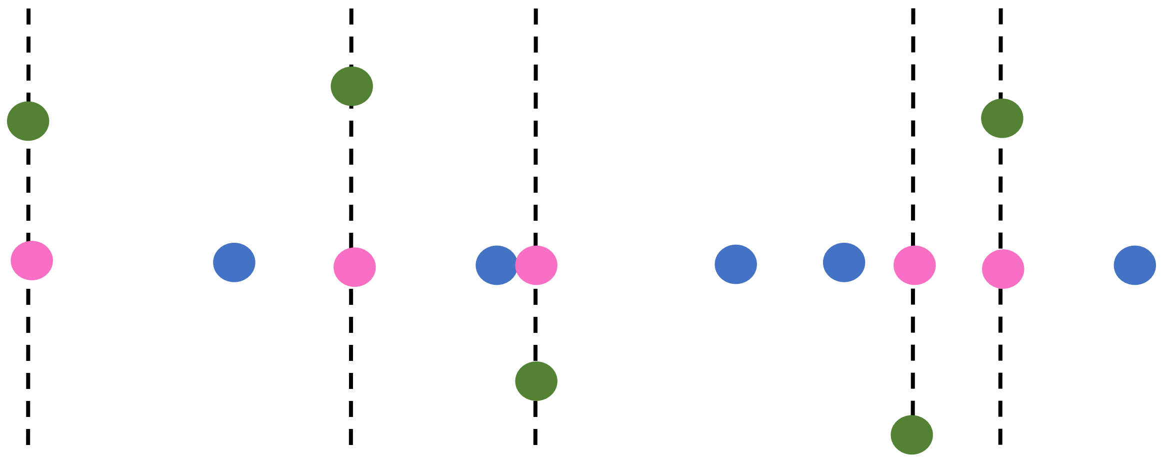

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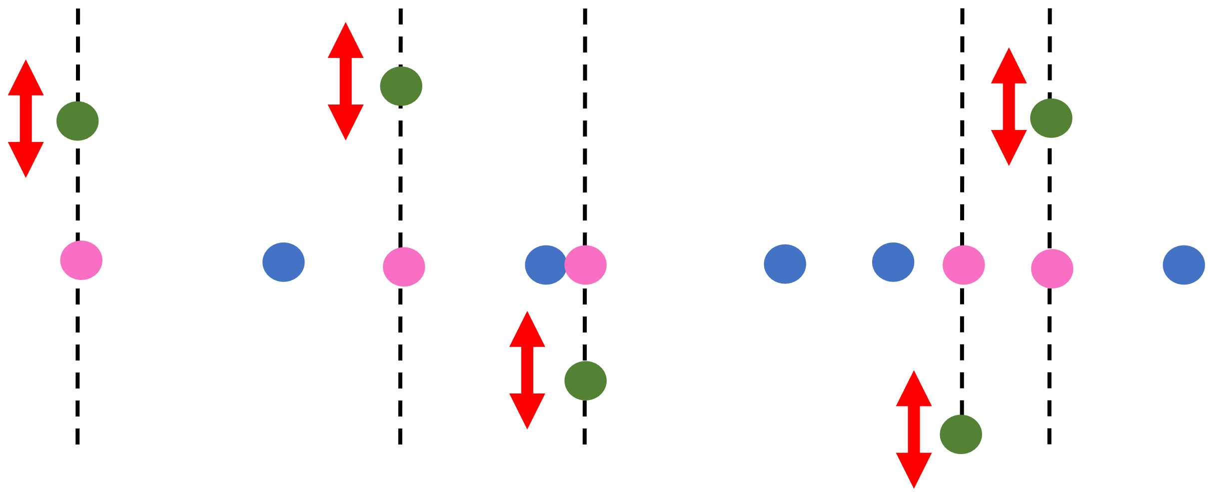

Alternative perspective:
Low-Rank Optimization

Notation and reformulation

$$Z = \begin{pmatrix} z_1^\top \\ \vdots \\ z_n^\top \end{pmatrix} \in \mathbb{R}^{n\times\ell}, \qquad Z_* = \begin{pmatrix} {z_1^*}^\top \\ \vdots \\ {z_n^*}^\top \end{pmatrix} \in \mathbb{R}^{n\times\ell}$$

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Gram $\to$ EDM (euclidean distance matrix)

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$$[\Delta(Y)]_{ij} := Y_{ii} + Y_{jj} - 2Y_{ij}$$

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$$\min \|\Delta(ZZ^\top - Z_* Z_*^\top)\|^2 \text{ over } Z \in \mathbb{R}^{n\times\ell}$$

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- If $k = n$, problem is convex (1-critical points are global mins)

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Burer-Monteiro factorization!

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- Conclusion: Landscape benign if $k = n$

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Δ does not satisfy RIP! Δ has RIP-condition-number n

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New "general" theorem: If Γ is completely positive, contractive, and satisfies

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E.g., $\Gamma(Y) = \sum_{i=1}^{N} a_i a_i^\top \left(a_i^\top Y a_i\right)$ with $a_i \in \mathbb{R}^n$

Takeaways

Summary:

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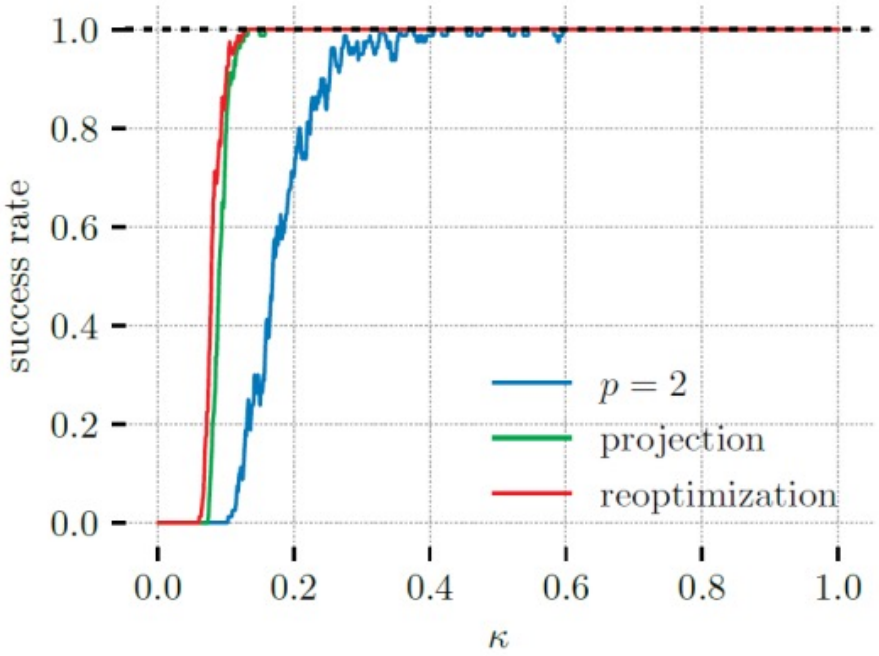
Conceptual takeaways:

- Low-dimensional nonconvex relaxations (cheap and often work!)
- Going beyond RIP: structured “perturbations”

Open questions

- **Conjecture [arbitrary GT]**: Relaxing to $k = \ell + 1$ is enough.
- **Conjecture [isotropic GT]**: Relaxing is not necessary.

- Many other localization problems (trajectory localization, inverse kinemetics, …)
- Incomplete graphs (random, expanders, …)



Conclusion

Conceptual takeaways:

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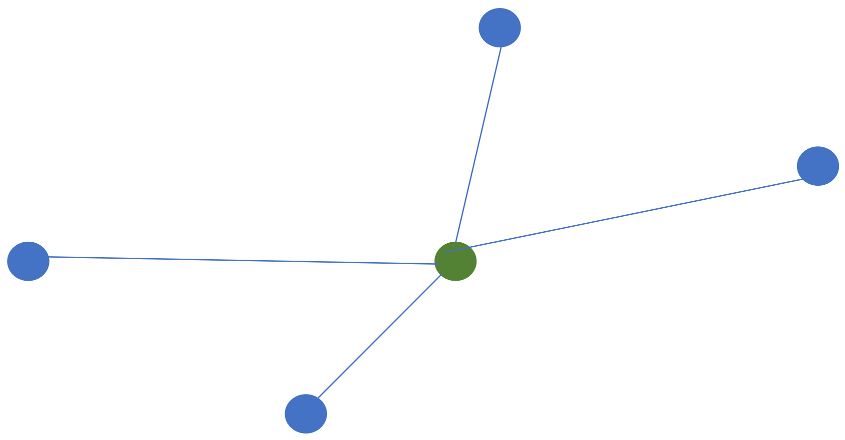
- Incomplete graphs (random, expanders, …)
- Many other localization problems (trajectory localization, inverse kinemetics, …)
- More general theory to analyze landscapes?

Appendix

SNL with landmarks

$$\min \sum_i \left(\|z - z_i\|^2 - d_i^2\right)^2, \qquad d_i = \|z^* - z_i^*\|$$

over $z \in \mathbb{R}^\ell$



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Landscape is not benign in general.

SNL with landmarks

$$\min_{\text{over } z \in \mathbb{R}^\ell} \sum_i \left(\|z - z_i\|^2 - d_i^2 \right)^2, \qquad d_i = \|z^* - z_i^*\|$$

Landscape is not benign in general.

Proposition: If relax to $k = \ell + 1$, the landscape is benign.

Hubs

Theorem [isotropic GT]: If graph is **nearly complete**, ground truth points are isotropic and iid, and relax to

$$k \approx \ell \log(n)\,,$$

then every 2-critical point is the ground truth.

The **hub** of a graph is the set of vertices which are connected to all other vertices.

$$H = \text{size of hub}$$

Theorem [isotropic GT]: If ground truth points are isotropic and iid, and relax to

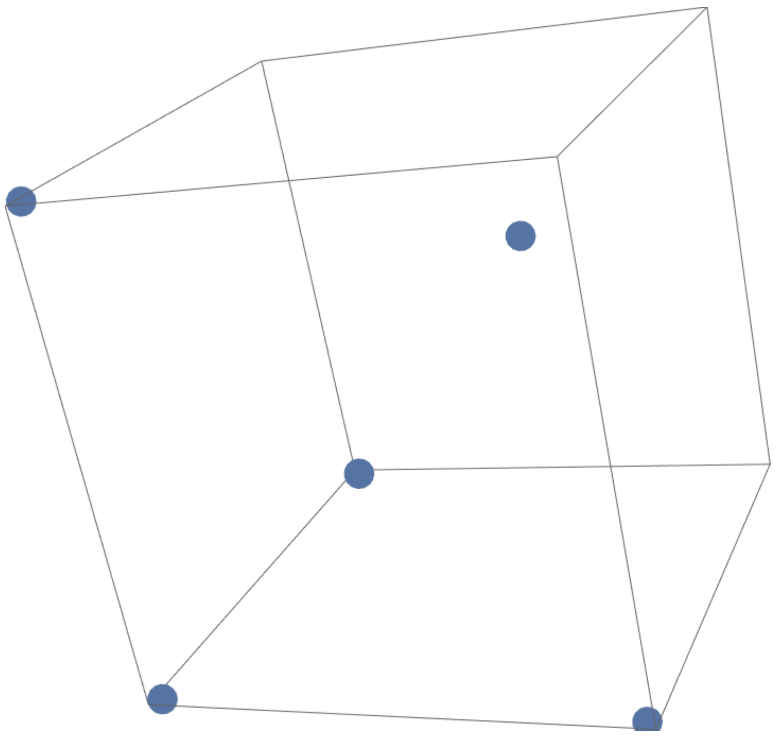
$$k \approx \text{poly}(n - H)\ell \log(n)\,,$$

then every 2-critical point is the ground truth.

Counterexamples

Minima number of points to have spurious local minima?

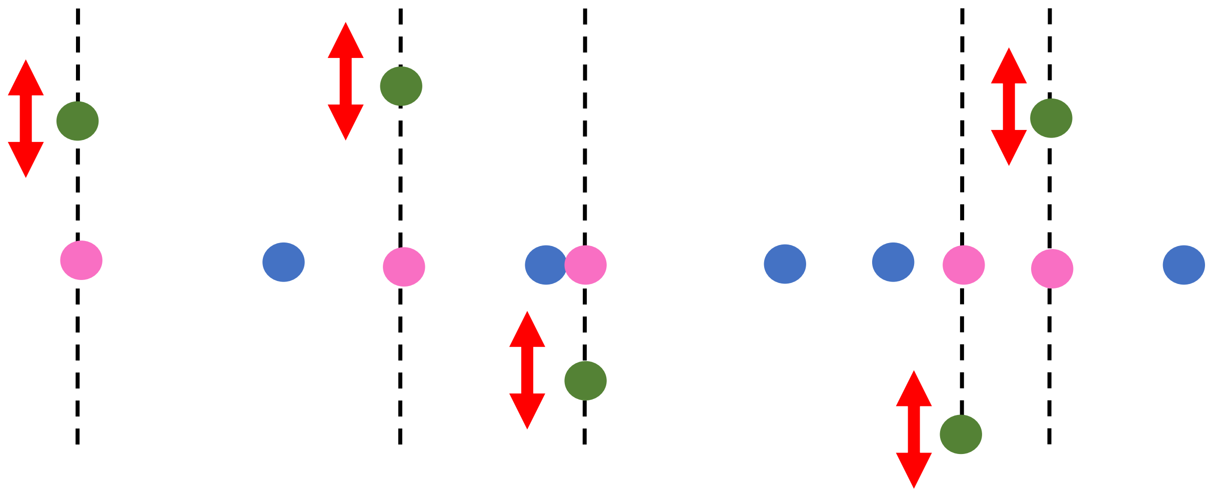
$$n = \ell + 2 \text{ (for } \ell \geq 5)$$



Some ideas from the proof

If relax enough, many ways to perturb this way

Use **eigenvalue interlacing** to argue that a good one exists, if relax enough



Some ideas from the proof

If relax enough, many ways to perturb this way

Use **eigenvalue interlacing** to argue that a good one exists, if relax enough

For **isotropic GT,** $k \approx \ell \log(n)$, similar descent direction

Randomize over descent directions (instead of eigenvalue interlacing)

Can we apply Kirwan convexity, or similar?

$$\min \|\Delta(ZZ^\top - Z_* Z_*^\top)\|^2 \text{ over } Z \in \mathbb{R}^{n\times\ell}$$

$\min \|ZZ^\top - Z_* Z_*^\top\|^2$ over $Z \in \mathbb{R}^{n\times\ell}$ (with $\operatorname{trace}(ZZ^\top) = 1$)

- Kirwan: $K = U(n)$ acts on projective space $\mathbb{P}\left(\mathbb{C}^{n\times\ell}\right)$

No index-1 critical points if relax to $k = \ell + 2$?

- Seems to be a common phenomenon when relaxing dimension

[Index = number of negative eigenvalues of Hessian]